NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2019	HSC Trial Examination

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 17)

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

NAME:____________________ **TEACHER:**____________________

STUDENT NUMBER: | | | | | | | | |

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/15	/15	/15	/15	/15	/100

BLANK PAGE

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

---

1 What is the value of $\left(2+i^5\right)\left(i^2-i^3\right)$?

(A) $5-5i$

(B) $7-i$

(C) $-1+3i$

(D) $-3+i$

2 Which expression is equal to $\int x^2 \cos x \, dx$?

(A) $x^2 \sin x + \int 2x \sin x \, dx$

(B) $x^2 \sin x - \int 2x \sin x \, dx$

(C) $2x \sin x - \int x^2 \sin x \, dx$

(D) $2x \sin x + \int x^2 \sin x \, dx$

3 A directrix of an ellipse has the equation $x = \dfrac{25}{4}$ and one of its foci has the coordinates $\left(-4,0\right)$. What is the equation of the ellipse?

(A) $\dfrac{x^2}{5}+\dfrac{y^2}{3}=1$

(B) $\dfrac{x^2}{3}+\dfrac{y^2}{5}=1$

(C) $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$

(D) $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$

4 Given the substitution $x = \pi - y$, which of the following is equal to $\int_0^{\pi} x \sin x \, dx$?

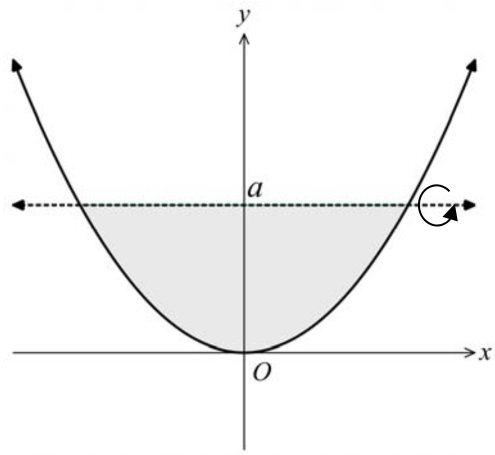
(A) $\int_{-\pi}^{\pi} \sin x \, dx$

(B) $\frac{\pi}{2}\int_0^{\pi} \sin x \, dx$

(C) $\pi\int_0^{\pi} \sin x \, dx$

(D) $\int_0^{\pi} \sin x \, dx$

5 The region bounded by the parabola $x^2 = 4ay$ and the line $y = a$ is rotated about the line $y = a$ to form a solid.



Which expression represents the volume of the solid?

(A) $2\pi\int_0^{2a}\left(a - \frac{x^2}{4a}\right)^2 dx$

(B) $2\pi\int_0^{2a}\left(a^2 - \left(\frac{x^2}{4a}\right)^2\right) dx$

(C) $\pi\int_0^{2a}\left(a - \frac{x^2}{4a}\right)^2 dx$

(D) $\pi\int_0^{2a}\left(a^2 - \left(\frac{x^2}{4a}\right)^2\right) dx$

6 Let e be the eccentricity of a conic, centred at the origin, with both foci on the x-axis. Which of the following is NOT true?

(A) If two ellipses have the same foci and directrices, then they have the same equation.

(B) If two hyperbolae have equal eccentricity, then they share the same asymptotes.

(C) For the hyperbola, as $e \to \infty$, the asymptotes approach the x-axis.

(D) For the ellipse, as $e \to 0$, the directrices move further away from the origin whilst the foci approach the origin.

7 Which complex number is a root of $z^6 + i = 0$?

(A) $-1-i$

(B) $-1+i$

(C) $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

(D) $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

8 If the tangent at the point $(2\sec\phi, 3\tan\phi)$ on the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x-y+4=0$, then what is a possible value of ϕ ?

(A) $45°$

(B) $60°$

(C) $30°$

(D) $75°$

9 Let the complex number z satisfy the equation $|z+4i|=3$. What are the greatest and least values of $|z+3|$?

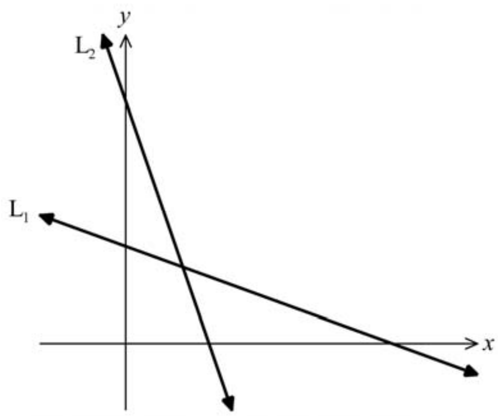
(A) 8 and 2

(B) 5 and 2

(C) 8 and 3

(D) 8 and 5

10 The diagram below shows the graphs of the straight lines L_1 and L_2, whose equations are $y=ax+b$ and $y=cx+d$ respectively.



Which of the following are true?

I. $c<a$
II. $d>b$
III. $ad>bc$

(A) I and II only

(B) I and III only

(C) II and III only

(D) I, II and III

Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hour 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Express $\dfrac{4+3i}{2-i}$ in the form $x+iy$, where x and y are real. **2**

(b) Consider the complex numbers $z=-1+\sqrt{3}i$ and $w=2\left(\cos\dfrac{\pi}{5}+i\sin\dfrac{\pi}{5}\right)$.

(i) Express z in modulus-argument form. **2**

(ii) Find the argument of $\dfrac{z}{w}$. **1**

(c) (i) Express $\dfrac{3x^2+x+6}{(x^2+9)(x-1)}$ in the form $\dfrac{Ax+B}{x^2+9}+\dfrac{C}{x-1}$. **2**

(ii) Hence find $\displaystyle\int\frac{3x^2+x+6}{\left(x^2+9\right)\left(x-1\right)}\,dx$. **2**

(d) The equation $|z-2|-|z+2|=\pm 2$ corresponds to a conic in the Argand diagram. Sketch the conic, showing any asymptotes, foci and directrices. **3**

(e) The polynomial $P(x)=x^4+3x^3-x^2-13x-10$ has a zero at $x=-2-i$.

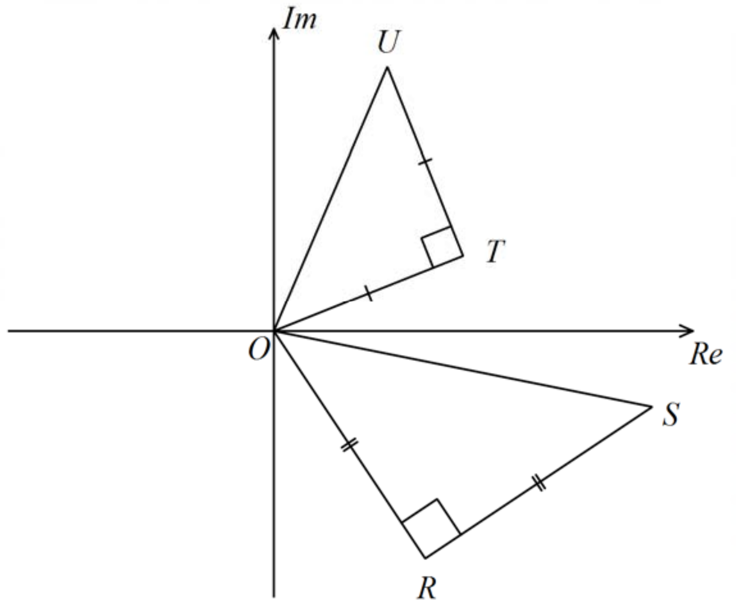
(i) Explain why $x=-2+i$ is also a zero. **1**

(ii) Fully factorise $P(x)$ over the real numbers. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a)



In the Argand diagram above, the points O, R, S, T and U correspond to the complex numbers $0, r, s, t$ and u respectively. The triangles ORS and OTU are right-angled isosceles triangles. Let $\omega = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$.

(i) Explain why $u = \sqrt{2}\,\omega t$. **1**

(ii) Show that $r = \dfrac{s}{\sqrt{2}\,\omega}$. **1**

(iii) Using complex numbers show that $\dfrac{SU}{RT} = \sqrt{2}$. **2**

(b) Let $I = \displaystyle\int \frac{\sin x}{\sin x + 2\cos x}\,dx$ and $J = \displaystyle\int \frac{\cos x}{\sin x + 2\cos x}\,dx$.

(i) Find $I + 2J$. **1**

(ii) Find $2I - J$. **1**

(iii) Hence, or otherwise, find $\displaystyle\int \frac{\sin x}{\sin x + 2\cos x}\,dx$. **2**

Question 12 continues on Page 9

Question 12 (continued)

(c) (i) Sketch the curve $f(x) = \dfrac{4x^2}{x^2 - 9}$ showing all intercepts and asymptotes. **2**

(ii) Hence sketch $|y| = f(x)$ on a separate number plane. **2**

(d) A relation is defined by the equation $\tan^{-1}(x^2) + \tan^{-1}(y^2) = \dfrac{\pi}{4}$.

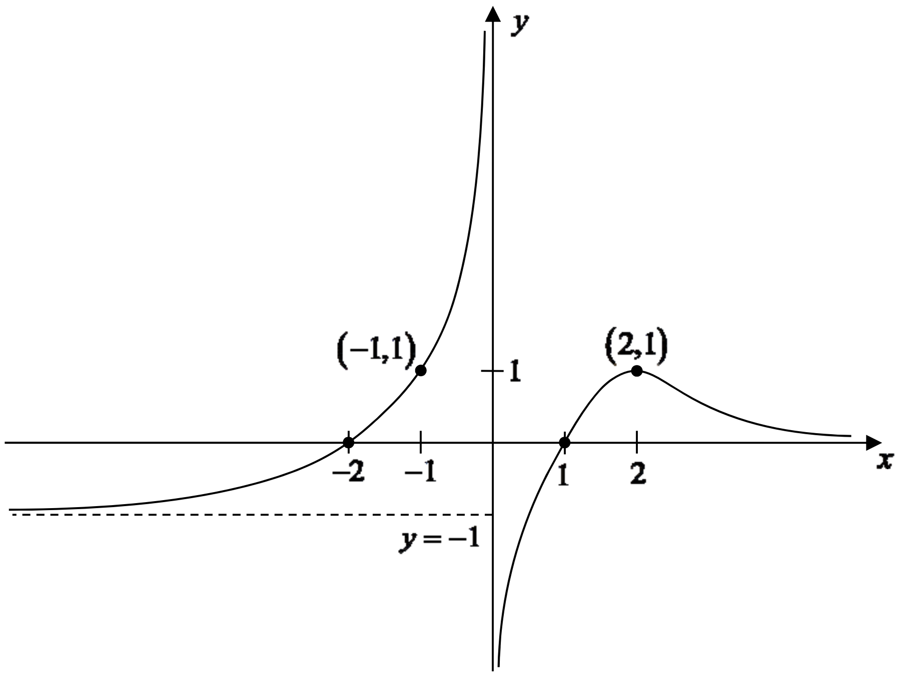
(i) Find $\dfrac{dy}{dx}$ in terms of x and y. **1**

(ii) Find the gradient of the curve at the point where $x = \dfrac{1}{\sqrt{2}}$ and $y < 0$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows a sketch of $y = f(x)$ with asymptotes at $x = 0$, $y = -1$ and $y = 0$. There is a maximum turning point at $(2,1)$ and the curve passes through $(-1,1)$.



Neatly sketch the graphs of the following showing all important information, including the coordinates of any new points which can be determined.

(i) $y^2 = f(x)$ **2**

(ii) $y = e^{f(x)}$ **2**

(b) (i) Prove that for any polynomial $P(x)$, if k is a zero of multiplicity r, then k is a zero of multiplicity $r-1$ of $P'(x)$. **1**

(ii) Given that the polynomial $P(x) = x^4 + 5x^3 + 9x^2 + 7x + 2$ has a zero of multiplicity 3, factorise $P(x)$. **3**

Question 13 continues on Page 11

Question 13 (continued)

(c) The cubic equation $x^3 + 4x + 3 = 0$ has roots α, β and γ.

(i) Find a polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence, or otherwise, find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

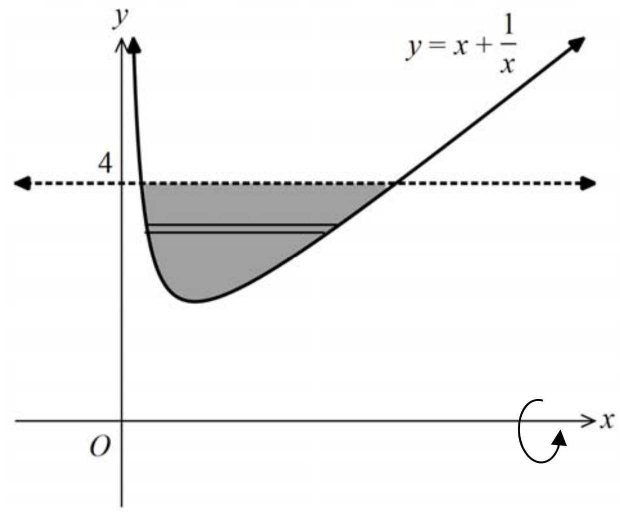
(d) A sequence is defined by $a_1 = 1$, $a_2 = 8$ and $a_{n+2} = a_{n+1} + 2a_n$ for all positive integers n. Use Mathematical Induction to prove that $a_n = 3 \times 2^{n-1} + 2(-1)^n$. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) (i) Let x be a positive real number. Show that $x + \dfrac{1}{x} \geq 2$. **1**

(ii) The region bounded by the curve $y = x + \dfrac{1}{x}$ and the line $y = 4$ is rotated about the x-axis. **4**



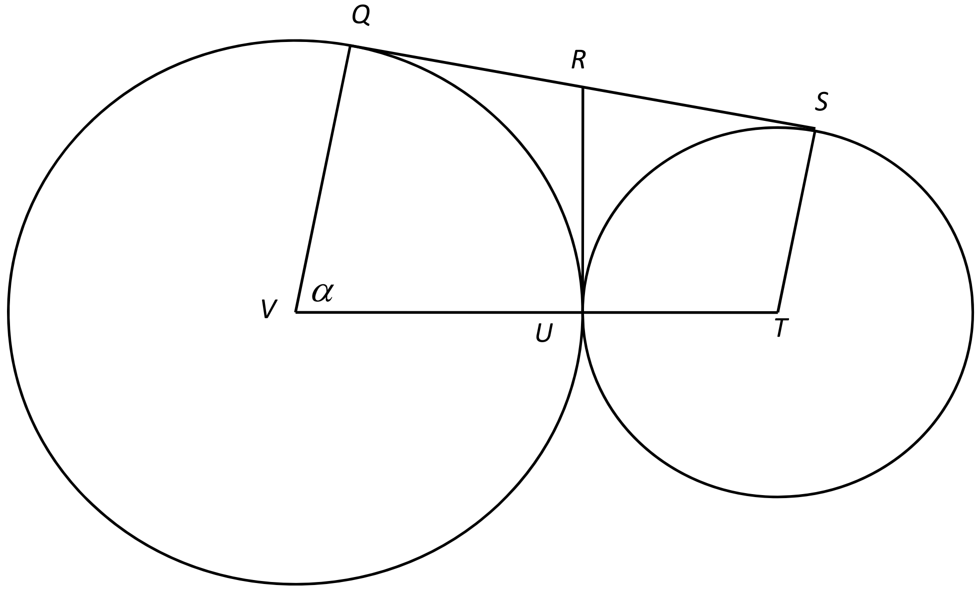
Use the method of cylindrical shells to show that the volume of the solid of revolution formed is $16\pi\sqrt{3}$ units^3.

(b) (i) Show that if x and y are positive and $x^3 + x^2 = y^3 - y^2$, then $x < y$. **2**

(ii) Show that if $0 < x \leq y - 1$, then $x^3 + x^2 < y^3 - y^2$. **2**

Question 14 continues on Page 13

(c) In the diagram, VUT is a straight line joining V and T, the centres of the circles. QS and RU are common tangents. Let $\angle QVU = \alpha$.



Copy the diagram into your answer booklet.

(i) Explain why $QRUV$ and $RSTU$ are cyclic quadrilaterals. **1**

(ii) Show that ΔSRU is similar to ΔQVU. **3**

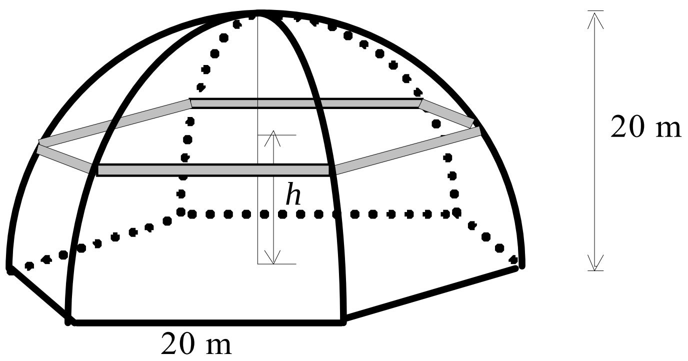
(iii) Show that QU is parallel to RT. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Determine $\int \cos^2 x \sin^7 x \, dx$. **3**

(b) A dome is sitting on a regular hexagonal base of side 20 metres. The height of the dome is also 20 metres. Each strut of the dome is a quarter of a circle with its centre at the centre of the hexagonal base.



A cross-sectional slice is taken parallel to the base of the dome.

(i) If the slice is h metres above the base, deduce that the length of each side is $\sqrt{400-h^2}$. **2**

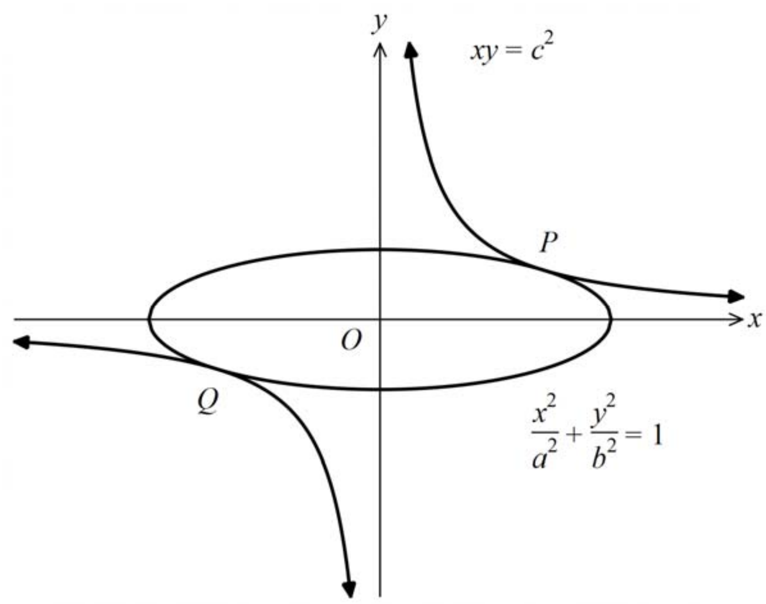
(ii) Show that the area of the cross-section is $A=\frac{3\sqrt{3}}{2}\left(400-h^2\right)$. **1**

(iii) Find the volume of the solid. **2**

Question 15 continues on Page 15

Question 15 (continued)

(c) The rectangular hyperbola $x = ct,\ y = \dfrac{c}{t}$, where $c > 0$, touches the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b > 0$, at points P and Q, where $P\left(cp, \dfrac{c}{p}\right)$ lies in the first quadrant.



(i) Explain why the equation $(bc)^2 t^4 - (ab)^2 t^2 + (ca)^2 = 0$ has roots $p,\ p,\ -p,\ -p$ where $p > 0$. **2**

(ii) Deduce that $p = \dfrac{a}{c\sqrt{2}}$ and $ab = 2c^2$. **2**

(iii) Show that if S and S' are the foci of the hyperbola $xy = c^2$, then the quadrilateral with vertices P, S, Q and S' has area $2c(a-b)$. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider $I_n = \int \frac{x^n}{\sqrt{a^2 - x^2}}\,dx, \quad n \geq 0$.

(i) Show that $nI_n = -x^{n-1}\sqrt{a^2 - x^2} + a^2(n-1)I_{n-2}$ where $n \geq 2$. **3**

(ii) Hence find $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$. **1**

(b) Let $P(x)$ be a polynomial of degree n, where n is odd.

It is known that $P(k) = \frac{k}{k+1}$ for $k = 0,\ 1,\ 2,\ ...,\ n$.

(i) $Q(x)$ is a polynomial such that $Q(x) = (x+1)P(x) - x$. Show that the zeroes of $Q(x)$ are $x = 0, 1, 2, ..., n$. **1**

(ii) Let A be the leading coefficient of $Q(x)$. Factor $Q(x)$, and show that $A = \frac{1}{1 \times 2 \times 3 \times ... \times n \times (n+1)} = \frac{1}{(n+1)!}$. **2**

(iii) Find $P(n+1)$. **1**

Question 16 continues on Page 17

Question 16 (continued)

(c) (i) Show that $x - \log_e(1 + x) > 0$ for $x > 0$. **2**

(ii) Hence show that $\sum_{k=1}^{n} \frac{1}{k} > \log_e(n + 1)$. **2**

(iii) Hence by considering $x + \log_e(1 - x)$, show that $\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \log_e 2$. **3**

End of paper

NORTH SYDNEY GIRLS HIGH SCHOOL
2019 HSC Trial Examination

Mathematics Extension 2- Solutions

1 What is the value of $(2+i^5)(i^2-i^3)$?

(A) $5-5i$

(B) $7-i$

(C) $-1+3i$

(D) $-3+i$

$(2+i^5)\,i^2(1-i)$
$= -(2+i)(1-i)$
$= -2-i+2i-1$
$= -3+i$

2 Which expression is equal to $\int x^2 \cos x \, dx$?

(A) $x^2 \sin x + \int 2x \sin x \, dx$

(B) $x^2 \sin x - \int 2x \sin x \, dx$

(C) $2x \sin x - \int x^2 \sin x \, dx$

(D) $2x \sin x + \int x^2 \sin x \, dx$

$u = x^2 \qquad v' = \cos x$
$u' = 2x \qquad v = \sin x$

$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

3 A directrix of an ellipse has the equation $x = \dfrac{25}{4}$ and one of its foci has the coordinates $(-4,0)$. What is the equation of the ellipse?

(A) $\dfrac{x^2}{5} + \dfrac{y^2}{3} = 1$

(B) $\dfrac{x^2}{3} + \dfrac{y^2}{5} = 1$

(C) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

(D) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

$\dfrac{a}{e} = \dfrac{25}{4}$, $\quad ae = 4$

$\dfrac{a}{\cancel{e}} \times a\cancel{e} = 25$

$a^2 = 25.$

Foci on the x-axis $\therefore$ C

4 Given the substitution $x = \pi - y$, which of the following is equal to $\int_0^{\pi} x \sin x \, dx$?

(A) $\int_{-\pi}^{\pi} \sin x \, dx$

(B) $\frac{\pi}{2}\int_0^{\pi} \sin x \, dx$

(C) $\pi\int_0^{\pi} \sin x \, dx$

(D) $\int_0^{\pi} \sin x \, dx$

$\int_0^{\pi} x\sin x \, dx = \int_{\pi}^{0} -(\pi-y)\sin(\pi-y)\,dy$

Let $x = \pi - y$, $dx = -dy$

when $x=0$, $y=\pi$; $x=\pi$, $y=0$

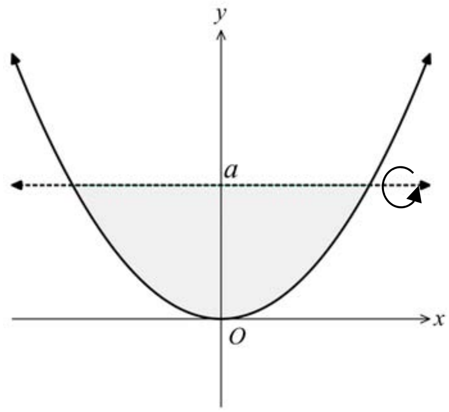
$= \int_0^{\pi} (\pi-y)\sin y \, dy$

$= \int_0^{\pi} \pi \sin y \, dy - \int_0^{\pi} y\sin y \, dy$

$2\int_0^{\pi} x\sin x \, dx = \int_0^{\pi} \pi\sin y \, dy$

$\therefore \int_0^{\pi} x\sin x \, dx = \frac{\pi}{2}\int_0^{\pi} \sin y \, dy$

5 The region bounded by the parabola $x^2 = 4ay$ and the line $y = a$ is rotated about the line $y = a$ to form a solid. Which expression represents the volume of the solid?



(A) $2\pi\int_0^{2a}\left(a - \frac{x^2}{4a}\right)^2 dx$

(B) $2\pi\int_0^{2a}\left(a^2 - \left(\frac{x^2}{4a}\right)^2\right) dx$

(C) $\pi\int_0^{2a}\left(a - \frac{x^2}{4a}\right)^2 dx$

(D) $\pi\int_0^{2a}\left(a^2 - \left(\frac{x^2}{4a}\right)^2\right) dx$

$\Delta A = \pi(a-y)^2$

$= \pi\left(a - \frac{x^2}{4a}\right)^2$

$y = \frac{x^2}{4a}$ $y = a$

$\frac{x^2}{4a} = a$

$V = \pi\int_{-2a}^{2a}\left(a - \frac{x^2}{4a}\right)^2 dx$

$x^2 = 4a^2$

$x = \pm 2a$

$= 2\pi\int_0^{2a}\left(a - \frac{x^2}{4a}\right)^2 dx.$

6 Let e be the eccentricity of a conic, centred at the origin, with both foci on the x-axis. Which of the following is NOT true?

(A) If two ellipses have the same foci and directrices, then they have the same equation.

Since $b^2 = a^2(1-e^2)$, if a and e are the same then b is also the same.

(B) If two hyperbolae have equal eccentricity, then they share the same asymptotes.

Since $\frac{b^2}{a^2} = e^2 - 1$, if e is the same then $\pm\frac{b}{a}$ is the same.

(C) For the hyperbola, as $e \to \infty$, the asymptotes approach the x-axis.

Since $\frac{b^2}{a^2} = e^2 - 1$, as $e \to \infty$ $\frac{b}{a} \to \infty$

$\therefore$ asymptotes do <u>not</u> approach the x-axis.

(D) For the ellipse, as $e \to 0$, the directrices move further away from the origin whilst the foci approach the origin.

As $e \to 0$, $\frac{a}{e} \to \infty$ and $ae \to 0$.

7 Which complex number is a root of $z^6 + i = 0$?

(A) $-1-i$

(B) $-1+i$

(C) $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

(D) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

$z^6 = \text{cis}\left(-\frac{\pi}{2} + 2k\pi\right)$

$z = \text{cis}\left(\frac{4k-1}{12}\pi\right)$ By De Moivre's Theorem

$= \text{cis}\left(-\frac{9\pi}{12}\right), \text{cis}\left(-\frac{5\pi}{12}\right), \text{cis}\left(-\frac{\pi}{12}\right), \text{cis}\left(\frac{3\pi}{12}\right), \text{cis}\left(\frac{7\pi}{12}\right), \text{cis}\left(\frac{11\pi}{12}\right)$

Now $\text{cis}\left(\frac{-9\pi}{12}\right) = \text{cis}\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

8 If the tangent at the point $(2\sec\phi, 3\tan\phi)$ on the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is parallel to $3x - y + 4 = 0$, then what is a possible value of ϕ ?

(A) 45°

(B) 60°

(C) 30°

(D) 75°

$\frac{dx}{d\phi} = 2\sec\phi\tan\phi$ $\quad \frac{dy}{d\phi} = 3\sec^2\phi$

$\frac{dy}{dx} = 3\sec^2\phi \times \frac{1}{2\sec\phi\tan\phi}$

$= \frac{3}{2\sin\theta}$

$\frac{3}{2\sin\theta} = 3 \Rightarrow \sin\phi = \frac{1}{2}$

$\phi = 30°$

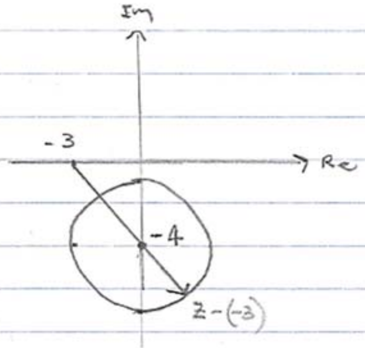
9 Let the complex number z satisfy the equation $|z+4i|=3$. What are the greatest and least values of $|z+3|$?

(A) 8 and 2

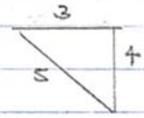
(B) 5 and 2

(C) 8 and 3

(D) 8 and 5



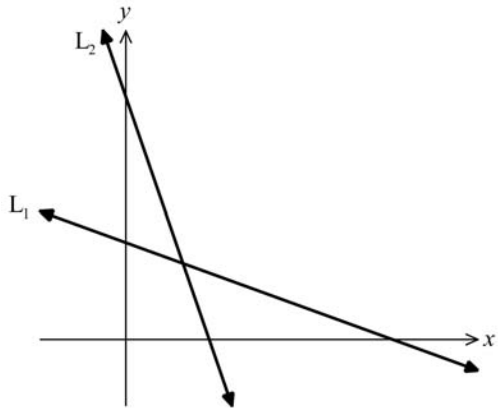
$|z-(-4i)|=3$



$\min|z-(-3)| = 5-3$
$= 2$

$\max|z-(-3)| = 2+6$
$= 8$

10 The diagram below shows the graphs of the straight lines L_1 and L_2, whose equations are $y=ax+b$ and $y=cx+d$ respectively.



Which of the following are true?

I. $c<a$
II. $d>b$
III. $ad>bc$

(A) I and II only

(B) I and III only

(C) II and III only

(D) I, II and III

y-intercept of L_2 > y-intercept of L_1 $\therefore d>b$
gradient of L_2 steeper than gradient of L_1 $c<a$

x-intercept of L_1 > x-intercept of L_2 $\frac{-b}{a} > \frac{-d}{c}$

$\frac{b}{a} < \frac{d}{c}$

since $a, c<0$ $bc<ad$

$\therefore ad>bc$

Question 11

(a) Express $\dfrac{4+3i}{2-i}$ in the form $x+iy$, where x and y are real. **2**

$$\frac{4+3i}{2-i} \times \frac{2+i}{2+i} = \frac{8+4i+6i-3}{4+1}$$

$$= \frac{5+10i}{5}$$

$$= 1+2i$$

(b) Consider the complex numbers $z=-1+\sqrt{3}i$ and $w=2\left(\cos\dfrac{\pi}{5}+i\sin\dfrac{\pi}{5}\right)$.

(i) Express z in modulus-argument form. **2**

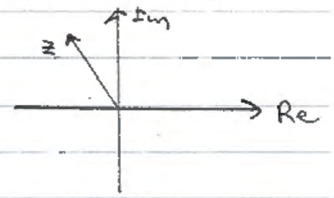
(ii) Find the argument of $\dfrac{z}{w}$. **1**

i) $|z| = \sqrt{(-1)^2+(\sqrt{3})^2}$

$= 2$

$\tan(\arg(z)) = -\sqrt{3}$

$\arg(z) = \dfrac{2\pi}{3}$



$z = 2\operatorname{cis}\dfrac{2\pi}{3}$

$= 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

ii) $\arg\left(\dfrac{z}{w}\right) = \dfrac{2\pi}{3} - \dfrac{\pi}{5}$

$= \dfrac{7\pi}{15}$

(c) (i) Express $\dfrac{3x^2+x+6}{(x^2+9)(x-1)}$ in the form $\dfrac{Ax+B}{x^2+9}+\dfrac{C}{x-1}$. **2**

(ii) Hence find $\displaystyle\int \frac{3x^2+x+6}{\left(x^2+9\right)\left(x-1\right)}\,dx$. **2**

i) $\dfrac{3x^2+x+6}{(x^2+9)(x-1)} = \dfrac{Ax+B}{x^2+9} + \dfrac{C}{x-1}$

$= \dfrac{(Ax+B)(x-1)+C(x^2+9)}{(x^2+9)(x-1)}$

$3x^2+x+6 = (Ax+B)(x-1)+C(x^2+9)$

sub $x=1$, $\quad 10 = 10C$

$C = 1$

equate coefficients of x^2, $\quad 3 = A + C \Rightarrow A = 2.$

sub $x=0$, $\quad 6 = -B + 9C$

$-3 = -B \Rightarrow B = 3$

$$\frac{3x^2+x+6}{(x^2+9)(x-1)} = \frac{2x+3}{x^2+9} + \frac{1}{x-1}$$

ii) $$\int \frac{3x^2+x+6}{(x^2+9)(x-1)}\,dx = \int \frac{2x\,dx}{x^2+9} + \int \frac{3}{x^2+9}\,dx + \int \frac{dx}{x-1}$$

$$= \log_e(x^2+9) + \tan^{-1}\frac{x}{3} + \log_e|x-1| + C$$

(d)	The equation $\lvert z-2\rvert - \lvert z+2\rvert = \pm 2$ corresponds to a conic in the Argand diagram. Sketch the conic, showing any asymptotes, foci and directrices.	3

foci at $(\pm 2, 0)$ since $PS - PS' = \pm 2a$ (alternate locus definition of hyperbola)

$a = 1$

$ae = 2$

$\therefore e = 2$

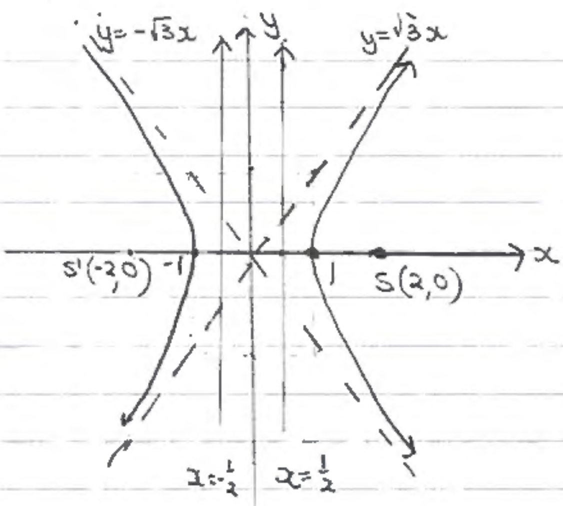
$b^2 = a^2(e^2-1)$

$b^2 = 4 - 1 \Rightarrow b = \sqrt{3}.$

directrices: $x = \pm \frac{a}{e}$

$x = \pm \frac{1}{2}$

asymptotes: $y = \pm\sqrt{3}\,x$



(e)	The polynomial $P(x) = x^4 + 3x^3 - x^2 - 13x - 10$ has a zero at $x = -2 - i$.	
	(i) Explain why $x = -2 + i$ is also a zero.	1
	(ii) Fully factorise $P(x)$ over the real numbers.	2

i) complex roots of polynomial with real coefficients occur in conjugate pairs.

ii) Let the roots be $\alpha, \beta, -2 \pm i$

sum of roots: $\alpha + \beta - 4 = -3$

$\alpha + \beta = 1$

product of roots: $5\alpha\beta = -10$

$\alpha\beta = -2$

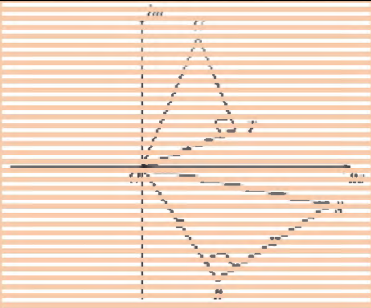
The equation $x^2 - x - 2 = 0$ has roots α, β.

$P(x) = (x - (-2+i))(x - (-2-i))(x^2 - x - 2)$

$= (x^2 + 4x + 5)(x - 2)(x + 1)$ over $\mathbb{R}$

Question 12

(a)



In the Argand diagram above, the points O, R, S, T and U correspond to the complex numbers $0, r, s, t$ and u respectively. The triangles ORS and OTU are right-angled isosceles triangles. Let $\omega = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$.

(i) Explain why $u = \sqrt{2}\,\omega t$. **1**

(ii) Show that $r = \dfrac{s}{\sqrt{2}\,\omega}$. **1**

(iii) Using complex numbers show that $\dfrac{SU}{RT} = \sqrt{2}$. **2**

i) To transform $\vec{OT}$ to $\vec{OU}$, rotate anti-clockwise by $\pi/4$ and scale by $\sqrt{2}$.
(Right-angled isosceles $\triangle$ lengths in ratio $1:1:\sqrt{2}$)
i.e. multiply by $\sqrt{2}\operatorname{cis}\pi/4 = \sqrt{2}w$

ii) To transform $\vec{OS}$ to $\vec{OR}$, rotate clockwise by $\pi/4$ and scale by $1/\sqrt{2}$.

i.e. multiply by $\frac{1}{\sqrt{2}}\operatorname{cis}(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}\operatorname{cis}(\pi/4)}$

iii) $\vec{SU} = \vec{OU} - \vec{OS}$
$= u - s$
$= \sqrt{2}wt - s$

$\vec{RT} = \vec{OT} - \vec{OR}$
$= t - r$
$= t - s/\sqrt{2}w$

$$\frac{|\vec{SU}|}{|\vec{RT}|} = \frac{|\sqrt{2}wt - s|}{\left|t - \frac{s}{\sqrt{2}w}\right|} = \frac{|\sqrt{2}wt - s|}{\left|\frac{\sqrt{2}wt - s}{\sqrt{2}w}\right|}$$

$$= \cancel{|\sqrt{2}wt - s|} \times \frac{|\sqrt{2}w|}{\cancel{|\sqrt{2}wt - s|}}$$

$$= |\sqrt{2}w|$$
$$= \sqrt{2}$$

(b) Let $I = \int \frac{\sin x}{\sin x + 2\cos x}\,dx$ and $J = \int \frac{\cos x}{\sin x + 2\cos x}\,dx$.

(i) Find $I + 2J$. **1**

(ii) Find $2I - J$. **1**

(iii) Hence, or otherwise, find $\int \frac{\sin x}{\sin x + 2\cos x}\,dx$. **2**

i) $I + 2J = \int \frac{\sin x + 2\cos x}{\sin x + 2\cos x}\,dx$

$= \int dx$

$= x + C$

ii) $2I - J = \int \frac{2\sin x - \cos x}{\sin x + 2\cos x}\,dx$

$= -\int \frac{\cos x - 2\sin x}{\sin x + 2\cos x}\,dx$

$= -\log_e|\sin x + 2\cos x| + C_1$

iii) $I + J + 2(2I - J) = x - 2\log_e|\sin x + 2\cos x| + C_2$

$5I = x - 2\log_e|\sin x + 2\cos x| + C_2$

$I = \frac{x}{5} - \frac{2}{5}\log_e|\sin x + 2\cos x| + C_3$

(c) (i) Sketch the curve $f(x) = \frac{4x^2}{x^2 - 9}$ showing all intercepts and asymptotes. **2**

(ii) Hence sketch $|y| = f(x)$ on a separate number plane. **2**

i) vertical asymptotes: $x = \pm 3$

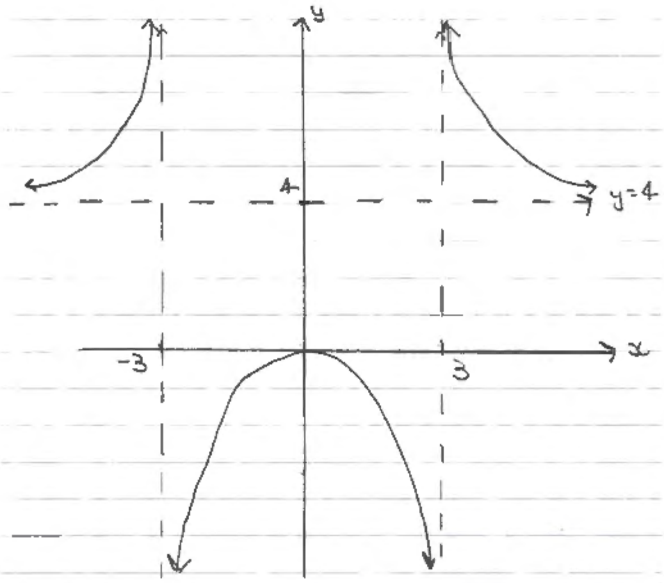
horizontal asymptote: $y = 4$

$f(-x) = f(x)$, $\therefore$ even function

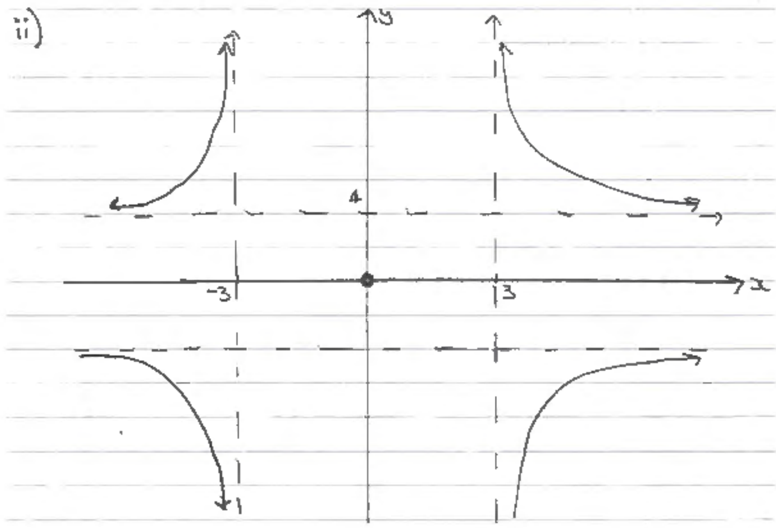
when $x = 0$, $y = 0$

crosses horizontal asymptote if $4 = \frac{4x^2}{x^2 - 9}$

$4x^2 - 36 = 4x^2$, No soln.



ii)



(d) A relation is defined by the equation $\tan^{-1}(x^2)+\tan^{-1}(y^2)=\frac{\pi}{4}$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. **1**

(ii) Find the gradient of the curve at the point where $x=\frac{1}{\sqrt{2}}$ and $y<0$. **2**

i) $\tan^{-1}(x^2)+\tan^{-1}(y^2)=\frac{\pi}{4}$

differentiating implicitly,

$$\frac{2x}{1+(x^2)^2}+\frac{2y}{1+(y^2)^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{1+x^4}\times\frac{1+y^4}{2y}$$

$$=\frac{-x(1+y^4)}{y(1+x^4)}$$

ii) when $x = \frac{1}{\sqrt{2}}$, $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(y^2) = \frac{\pi}{4}$

$$\tan^{-1}(y^2) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2}\right)$$

$$y^2 = \tan\left(\frac{\pi}{4} - \tan^{-1}(1/2)\right)$$

$$= \frac{1 - \frac{1}{2}}{1 + 1 \times \frac{1}{2}}$$

$$= \frac{1}{3}$$

$$y = -\frac{1}{\sqrt{3}} \quad (y < 0)$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{1}{\sqrt{2}}\left(1 + \frac{1}{9}\right)}{-\frac{1}{\sqrt{3}}\left(1 + \frac{1}{4}\right)} = \frac{\sqrt{3} \times \frac{10}{9}}{\sqrt{2} \times \frac{5}{4}}$$

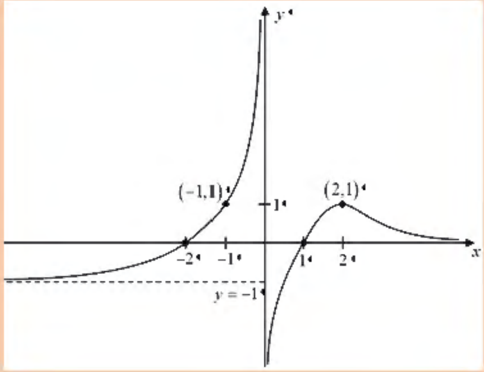
$$= \frac{10}{3\sqrt{3}} \times \frac{2\sqrt{2}}{5}$$

$$= \frac{4\sqrt{2}}{3\sqrt{3}} \quad \text{or} \quad \frac{8\sqrt{3}}{9\sqrt{2}}$$

Question 13

(a) The diagram shows a sketch of $y = f(x)$ with asymptotes at $x = 0$, $y = -1$ and $y = 0$.

There is a maximum turning point at $(2,1)$ and the curve passes through $(-1,1)$.

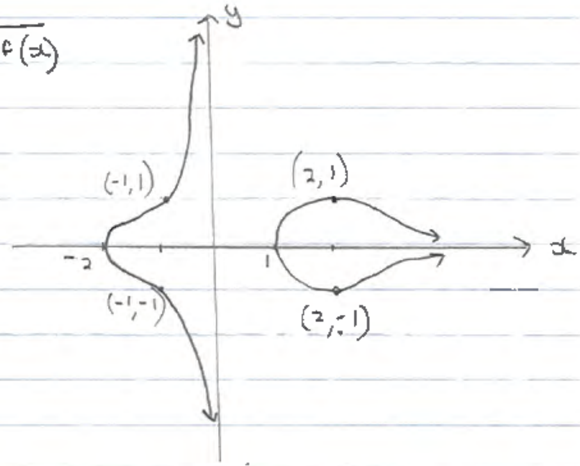


Neatly sketch the graphs of the following showing all important information, including the coordinates of any new points which can be determined.

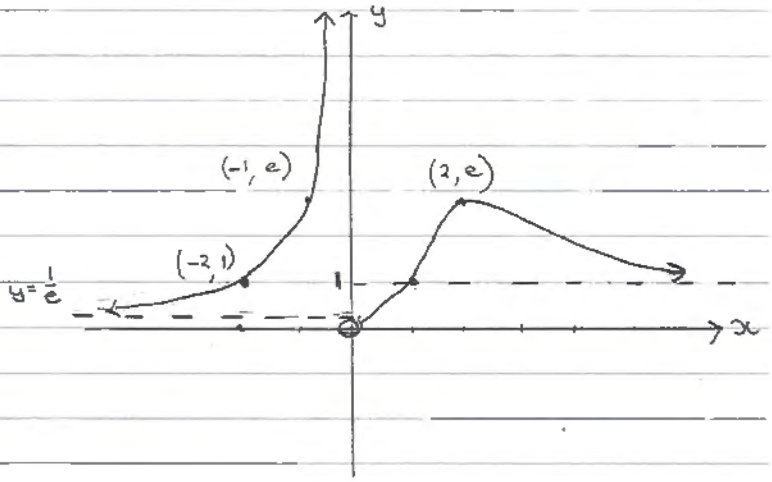
(i) $y^2 = f(x)$ **2**

(ii) $y = e^{f(x)}$ **2**

i) $y = \pm\sqrt{f(x)}$



ii)



(b)	(i)	Prove that for any polynomial $P(x)$, if k is a zero of multiplicity r, then k is a zero of multiplicity $r-1$ of $P'(x)$.	1
	(ii)	Given that the polynomial $P(x) = x^4 + 5x^3 + 9x^2 + 7x + 2$ has a zero of multiplicity 3, factorise $P(x)$.	3

i) Let $P(x) = (x-k)^r Q(x)$ where $Q(k) \neq 0$

$P'(x) = r(x-k)^{r-1} Q(x) + (x-k)^r . Q'(x)$

$= (x-k)^{r-1}\left[r Q(x) + (x-k) Q'(x) \right]$

$= (x-k)^{r-1} \left[R(x) \right]$ where $R(k) \neq 0$

$\therefore (x-k)^{r-1}$ is a zero of $P'(x)$ with multiplicity $(r-1)$.

ii) $P(x) = x^4 + 5x^3 + 9x^2 + 7x + 2$

$P'(x) = 4x^3 + 15x^2 + 18x + 7$

$P''(x) = 12x^2 + 30x + 18$

set $P''(x) = 0$, $12x^2 + 30x + 18 = 0$

$2x^2 + 5x + 3 = 0$

$(2x+3)(x+1) = 0$

$x = -\frac{3}{2}, -1$

$P(-1) = 1 - 5 + 9 - 7 + 2$

$= 0$

$\therefore -1$ is a triple root of $P(x) = 0$

Let the roots of $P(x) = 0$ be $-1, -1, -1, \alpha$

Sum of roots $\alpha - 3 = -5$

$\alpha = -2$

$\therefore P(x) = (x+2)(x+1)^3$

(c) The cubic equation $x^3 + 4x + 3 = 0$ has roots α, β and γ.

(i) Find a polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence, or otherwise, find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

i) Let $x^2 + 4x + 3 = 0$

Transforming the roots, $\alpha^2 = x$

$\alpha = \sqrt{x}$

$$(\sqrt{x})^3 + 4\sqrt{x} + 3 = 0$$
$$\sqrt{x}(x+4) = -3$$
$$x(x+4)^2 = 9$$
$$x^3 + 8x^2 + 16x - 9 = 0$$

This polynomial equation has roots α^2, β^2 & γ^2

ii) Let $A = \alpha^2$, $B = \beta^2$, $C = \gamma^2$

$$\alpha^4 + \beta^4 + \gamma^4 = A^2 + B^2 + C^2$$
$$A^2 + B^2 + C^2 = (A+B+C)^2 - 2(AB + AC + BC)$$
$$= (-8)^2 - 2(16) \quad \text{from i)}$$
$$= 32.$$

(d) A sequence is defined by $a_1 = 1$, $a_2 = 8$ and $a_{n+2} = a_{n+1} + 2a_n$ for all positive integers n. Use Mathematical Induction to prove that $a_n = 3 \times 2^{n-1} + 2(-1)^n$. **3**

Prove true for $n=1$ and $n=2$.

For $n=1$,

LHS $= 1$ RHS $= 3 \times 2^{1-1} + 2(-1)^1$

$= 1$

for $n=2$,

LHS $= 8$ RHS $= 3 \times 2^{2-1} + 2(-1)^2$

$= 6 + 2$

$= 8$

Assume true for $n=k$ and $n=k+1$

$a_k = 3 \times 2^{k-1} + 2(-1)^k$

$a_{k+1} = 3 \times 2^k + 2(-1)^{k+1}$

Prove true for $n = k+2$

i.e. R.T.P. $a_{k+2} = 3 \times 2^{k+2-1} + 2(-1)^{k+2}$

$= 3 \times 2^{k+1} + 2(-1)^{k+2}$

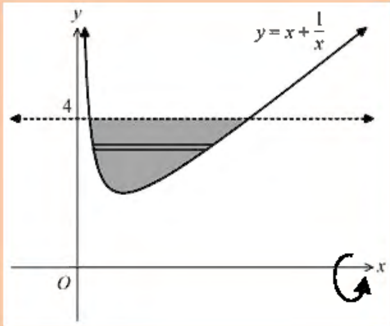
$$
\begin{aligned}
LHS &= a_{k+2} \\
&= a_{k+1} + 2a_k \\
&= 3 \times 2^{k} + 2(-1)^{k+1} + 2\left[3 \times 2^{k-1} + 2(-1)^k\right] \quad \text{(By assumption)} \\
&= 3 \times 2^k + 2(-1)^{k+1} + 3 \cdot 2^k + 4(-1)^k \\
&= 6 \times 2^k + 2(-1)(-1)^k + 4(-1)^k \\
&= 3 \times 2^{k+1} + 2(-1)^k \\
&= 3 \times 2^{k+1} + 2(-1)^{k+2} \\
&= RHS.
\end{aligned}
$$

$\therefore$ the statement is true by Mathematical Induction.

Question 14 (15 marks)

(a) (i) Let x be a positive real number. Show that $x + \frac{1}{x} \geq 2$. **1**

(ii) The region bounded by the curve $y = x + \frac{1}{x}$ and the line $y = 4$ is rotated about the x-axis. **4**



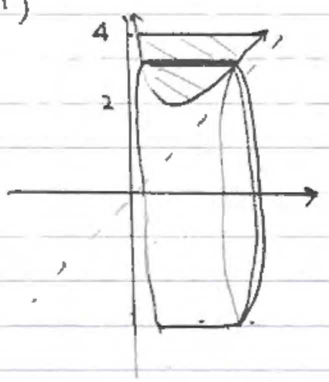
Use the method of cylindrical shells to show that the volume of the solid of revolution formed is $16\pi\sqrt{3}$ units^3.

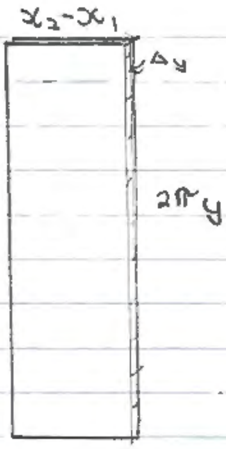
i) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$

$x + \frac{1}{x} - 2 \geq 0$

$x + \frac{1}{x} \geq 2.$

ii)





$y = x + \frac{1}{x}$

$xy = x^2 + 1$

$x^2 - xy + 1 \; 0$

$x = \frac{-y \pm \sqrt{(-y)^2 - 4}}{2}$

$= \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\therefore x_2 - x_1 = \frac{y + \sqrt{y^2 - 4}}{2} - \frac{y - \sqrt{y^2 - 4}}{2}$

$\Delta A = \sqrt{y^2-4} \times 2\pi y$

$\Delta V = 2\pi y\sqrt{y^2-4}\ \Delta y$

$$V = \lim_{\Delta y \to 0} \sum_{y=2}^{4} 2\pi y \sqrt{y^2-4}\ \Delta y$$

$$= \pi \int_2^4 2y\sqrt{y^2-4}\ dy.$$

$$= \pi \left[\frac{2(y^2-4)^{3/2}}{3} \right]_2^4$$

$$= \frac{2\pi}{3}\left(12^{3/2} - 0\right)$$

$$= \frac{2\pi}{3}\left(2\sqrt{3}\right)^3$$

$$= \frac{2\pi}{\cancel{3}} \times 8 \times \cancel{3}\sqrt{3}$$

$$= 16\pi\sqrt{3} \text{ units}^3$$

(b)	(i)	Show that if x and y are positive and $x^3 + x^2 = y^3 - y^2$, then $x < y$.	**2**
	(ii)	Show that if $0 < x \le y - 1$, then $x^3 + x^2 < y^3 - y^2$.	**2**

i) $x^3 + x^2 = y^3 - y^2$

$x^3 + x^2 < y^3 \qquad (y^2 > 0)$

$x^3 < y^3 \qquad (y^3 > 0)$

$x < y \qquad (x, y > 0)$

ii) $0 < x \le (y-1)$

squaring $0 < x^2 \le y^2 - 2y + 1$ ①

cubing $0 < x^3 \le (y-1)^3$

$= y^3 - 3y^2 + 3y - 1$ ②

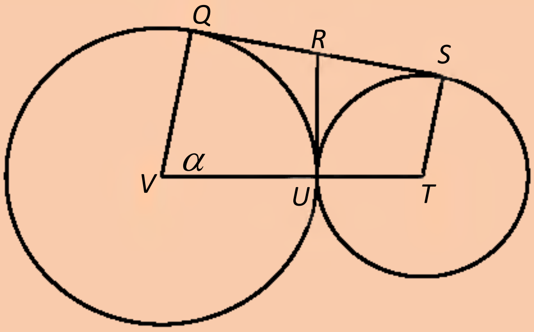
① + ② $x^3 + x^2 \le y^3 - 2y^2 + y$

$= y^3 - y^2 - (y^2 - y)$

$= y^3 - y^2 - y(y-1)$

$< y^3 - y^2 \qquad y > 0$ and $y - 1 > 0$

(c) In the diagram, VUT is a straight line joining V and T, the centres of the circles. QS and RU are common tangents. Let $\angle QVU = \alpha$.

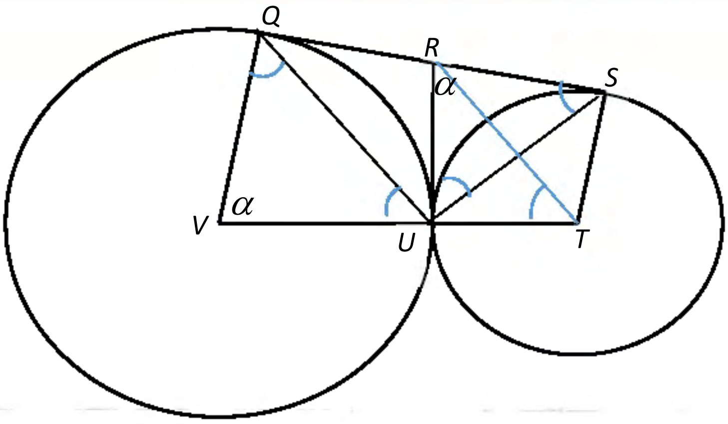


Copy the diagram into your answer booklet.

(i) Explain why $QRUV$ and $RSTU$ are cyclic quadrilaterals. **1**

(ii) Show that ΔSRU is similar to ΔQVU. **3**

(iii) Show that QU is parallel to RT. **2**



i) In each case a pair of opposite angles are equal to 90°. (tangent perpendicular to radius)

∴ QRUV and RSTU are cyclic quadrilaterals. (opposite angles are supplementary.)

ii) In ΔSRU and ΔQVU

$\angle QVU = \angle SRU$ (exterior angle of a cyclic quadrilateral)

$= \alpha$

$QV = VU$

$\angle VQU = \angle QUV$ (angles opposite equal sides)

$\angle QUV = \dfrac{180 - \alpha}{2}$

$RU = US$ (tangents from external point)

$\therefore \angle RSU = \angle RUS$ (angles opposite equal sides)

$\angle RSU = \dfrac{180 - \alpha}{2}$

$\therefore \angle VUQ = \angle RSU = \dfrac{180 - \alpha}{2}$

$\therefore \Delta SRU \parallel\!\parallel \Delta QVU$ (equiangular)

iii) In RSTU, RS = RS (tangents from external point)

UT = TS (radii)

RSTU is a kite (two pairs of adjacent sides equal)

$\angle UTS = 180 - \alpha$ (opposite angle of cyclic quad.)

$\angle RTU = \frac{180-\alpha}{2}$ (diagonal bisects vertex angle)

$\therefore \angle QUV = \angle RTU = \frac{180-\alpha}{2}$

QU ∥ RT (corresponding angles equal)

iii) ALTERNATE SOLUTION

$\angle VUQ = \angle RSU$ (shown in part ii)

construct RT

$\angle RSU = RTU$

$\therefore \angle RTU = \angle QUV$

QU ∥ RT (corresponding angles are equal)

Question 15

(a) Determine $\int \cos^2 x \sin^7 x \, dx$. **3**

a) $\int \cos^2 x \sin^7 x \, dx = \int \cos^2 x (\sin^2 x)^3 \sin x \, dx$

$= \int \cos^2 x (1 - \cos^2 x)^3 \sin x \, dx$

Let $u = \cos x$

$-du = \sin x \, dx$

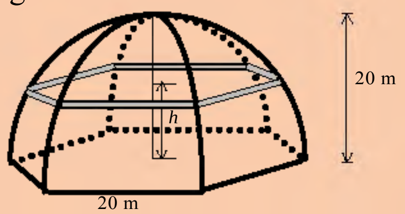
$= -\int u^2 (1-u^2)^3 \, du$

$= -\int (u^2 - 3u^4 + 3u^6 - u^8) \, du$

$= -\frac{u^3}{3} + \frac{3u^5}{5} - \frac{3u^7}{7} + \frac{u^9}{9} + C$

$= -\frac{1}{3}\cos^3 x + \frac{3}{5}\cos^5 x - \frac{3}{7}\cos^7 x + \frac{1}{9}\cos^9 x + C$

(b) A dome is sitting on a regular hexagonal base of side 20 metres. The height of the dome is also 20 metres. Each strut of the dome is a quarter of a circle with its centre at the centre of the hexagonal base.



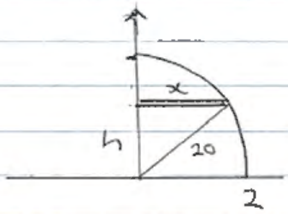
A cross-sectional slice is taken parallel to the base of the dome.

(i) If the slice is h metres above the base, deduce that the length of each side is $\sqrt{400-h^2}$. **2**

(ii) Show that the area of the cross-section is $A = \frac{3\sqrt{3}}{2}\left(400-h^2\right)$. **1**

(iii) Find the volume of the solid. **2**

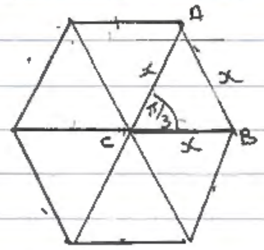
i)



$x^2 = 20^2 - h^2$ (Pythagoras Theorem)

$x = \sqrt{400-h^2}$

Cross-section



$\triangle ABC$ is equilateral triangle.

$\therefore$ side length of hexagon is x.

ii) Area of hexagon:

$$A = 6 \times \frac{1}{2} \times x \times x \times \sin\frac{\pi}{3}$$

$$= 3x^2 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}(400 - h^2)$$

iii) $\Delta V = A \Delta h$

$$V = \lim_{\Delta h \to 0} \sum_{h=0}^{20} \frac{3\sqrt{3}}{2}(400 - h^2)\Delta h$$

$$= \int_0^{20} \frac{3\sqrt{3}}{2}(400 - h^2)\, dh$$

$$= \frac{3\sqrt{3}}{2}\left[400h - \frac{h^3}{3}\right]_0^{20}$$

$$= \frac{3\sqrt{3}}{2}\left(8000 - \frac{8000}{3}\right)$$

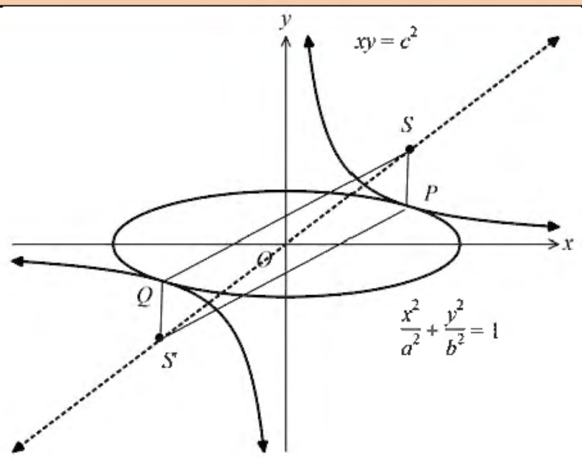
$$= 8000\sqrt{3} \text{ units}^3$$

(c) The rectangular hyperbola $x = ct,\ y = \frac{c}{t}$, where $c > 0$, touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, at points P and Q, where $P\left(cp, \frac{c}{p}\right)$ lies in the first quadrant.

(i) Explain why the equation $(bc)^2 t^4 - (ab)^2 t^2 + (ca)^2 = 0$ has roots $p,\ p,\ -p,\ -p$ where $p > 0$. **2**

(ii) Deduce that $p = \frac{a}{c\sqrt{2}}$ and $ab = 2c^2$. **2**

(iii) Show that if S and S' are the foci of the hyperbola $xy = c^2$, then the quadrilateral with vertices P, S, Q and S' has area $2c(a - b)$. **3**



i) $x = ct \quad y = \frac{c}{t}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

solving simultaneously for pts of intersection.

$$\frac{c^2t^2}{a^2} + \frac{c^2}{b^2t^2} = 1 \qquad \times\ a^2b^2t^2$$

$$b^2c^2t^4 + a^2c^2 = a^2b^2t^2$$
$$(bc)^2t^4 - (ab)^2t^2 + (ac)^2 = 0$$

This equation has 2 double roots as the ellipse and hyperbola are tangential to each other at the pts of contact.

Also by the symmetry of the ellipse & the hyperbola, P & Q are at opposite ends of a diameter, therefore their parameter values are opposite of each other.
Hence the roots are $p, p, -p, -p$ where p is the the parameter value at P.

ii) Sum of roots in pairs:
$$(p)(p) + (p)(-p) + (p)(-p) + p(-p) + p(-p) + (-p)(-p)$$

$$\therefore \quad -2p^2 = -\frac{(ab)^2}{(bc)^2}$$

$$p^2 = \frac{a^2}{2c^2}$$

$$\boxed{p = \frac{a}{c\sqrt{2}}} \quad (p>0) \quad ①$$

product of roots:
$$p \times p \times (-p) \times (-p) = \frac{(ca)^2}{(bc)^2}$$

$$p^4 = \frac{a^2}{b^2} \qquad ②$$

using ① & ②

$$p^4 = \frac{a^4}{c^4 \times 4} = \frac{a^2}{b^2} \qquad \div a^2$$

$$\frac{a^2}{4c^4} = \frac{1}{b^2}$$

$$a^2b^2 = 4c^4$$
$$ab = 2c^2 \qquad (ab > 0)$$

$$\therefore p = \frac{a}{c\sqrt{2}} \qquad ab = 2c^2$$

iii) $SPS'Q$ is a parallelogram (by symmetry)

Area of $SPS'Q = 2 \times$ Area $\triangle SPS'$

Perpendicular distance from P to SS' $= \dfrac{\left|cp - \frac{c}{p}\right|}{\sqrt{1^2+1^2}}$

$\downarrow$

$x - y = 0$

$$= \frac{\left|cp - \frac{c}{p}\right|}{\sqrt{2}}$$

$SS' = 4c$

$$\text{Area } \triangle SPS' = \frac{1}{2} \times \frac{1}{\sqrt{2}} \left|cp - \frac{c}{p}\right| \times 4c$$
$$= \sqrt{2}\, c^2 \left|p - \frac{1}{p}\right|$$
$$= \sqrt{2}\, c^2 \times \left(\frac{a}{c\sqrt{2}} - \frac{c\sqrt{2}}{a}\right) \qquad \left(\text{since } a > \frac{c}{\sqrt{2}} \text{ from diagram}\right)$$
$$= \sqrt{2}\, c^2 \left(\frac{a^2 - 2c^2}{ac\sqrt{2}}\right)$$
$$= \frac{c}{a}\left(a^2 - 2c^2\right)$$
$$= \frac{c}{a}\left(a^2 - ab\right) \qquad \text{from ii)}$$
$$= c(a - b)$$

$\therefore$ Area $SPS'Q = 2 \times$ Area $\triangle SPS'$
$= 2c(a - b)$

Question 16.

(a) Consider $I_n = \int \frac{x^n}{\sqrt{a^2 - x^2}} dx, \quad n \geq 0$.

(i) Show that $nI_n = -x^{n-1}\sqrt{a^2 - x^2} + a^2(n-1)I_{n-2}$ where $n \geq 2$. **3**

(ii) Hence find $\int \frac{x^2}{\sqrt{16 - x^2}} dx$. **1**

i) $I_n = \int \frac{x^n}{\sqrt{a^2-x^2}} dx \qquad n \geq 0$

$u = x^{n-1} \qquad v' = \frac{x}{\sqrt{a^2-x^2}}$

$u' = (n-1)x^{n-2} \qquad v = -\frac{2}{2}\sqrt{a^2-x^2}$

$= -\sqrt{a^2-x^2}$

$I_n = -x^{n-1}\sqrt{a^2-x^2} + (n-1)\int x^{n-2}\sqrt{a^2-x^2}\, dx$ (by parts)

$= -x^{n-1}\sqrt{a^2-x^2} + (n-1)\int x^{n-2}\frac{(a^2-x^2)}{\sqrt{a^2-x^2}} dx$

$= -x^{n-1}\sqrt{a^2-x^2} + (n-1)a^2\int \frac{x^{n-2}}{\sqrt{a^2-x^2}} dx - (n-1)\int \frac{x^n}{\sqrt{a^2-x^2}} dx$

$= -x^{n-1}\sqrt{a^2-x^2} + (n-1)a^2 I_{n-2} - (n-1)I_n$

ii) $\int \frac{x^2}{\sqrt{16-x^2}} dx = I_2$ (where $a=4$)

$2I_2 = -x\sqrt{16-x^2} + a^2(2-1)I_0$ from part i)

$= -x\sqrt{16-x^2} + 16\int \frac{1}{\sqrt{16-x^2}} dx$

$I_2 = -\frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\frac{x}{4} + C$

(b) Let $P(x)$ be a polynomial of degree n, where n is odd.

It is known that $P(k)=\dfrac{k}{k+1}$ for $k = 0,\ 1,\ 2,\ ...,\ n$.

(i) $Q(x)$ is a polynomial such that $Q(x)=\left(x+1\right)P(x)-x$. Show that the zeroes of $Q(x)$ are $x = 0, 1, 2,..., n$. **1**

(ii) Let A be the leading coefficient of $Q(x)$. Factor $Q(x)$, and show that $A=\dfrac{1}{1\times2\times3\times\ldots\times n\times\left(n+1\right)}=\dfrac{1}{(n+1)!}$. **2**

(iii) Find $P(n+1)$. **1**

i) sub. $x=k$

$Q(k)=(k+1)P(k)-k$

$=\cancel{(k+1)}\times\dfrac{k}{\cancel{k+1}}-k$ where $k=0,1,2\ldots n$

$=0$

$\therefore$ the zeroes of $Q(k)$ are $x=0,1,2\ldots n$ (by the factor theorem)

ii) $Q(x)=(x+1)P(x)-x$ ①

$Q(x)=Ax(x-1)(x-2)\ldots(x-n)$ ②

sub $x=-1$ in ②

$Q(-1)=-A(-2)(-3)\ldots(-1-n)$

$=(-1)A(-2)(-3)\ldots(-(n+1))$

$=(-1)^{n+1}A\times(n+1)!$

$=A\times(n+1)!$ * (since $n+1$ is even, $(-1)^{n+1}=1$)

sub $x=-1$ in ①

$Q(-1)=(-1+1)P(-1)+1$

$=1.$ **

Equating * and **

$\therefore A(n+1)!=1$

$A=\dfrac{1}{(n+1)!}$

iii) sub $x = n+1$

$$Q(n+1) = (n+1+1) \times P(n+1) - (n+1)$$
$$= (n+2)\,P(n+1) - (n+1) \quad (A)$$

$$Q(n+1) = \frac{1}{(n+1)!}(n+1)(n+1-1)(n+1-2) \times \dots \times (n+1-n)$$

$$= \frac{1}{(n+1)!}(n+1)(n)(n-1) \times \dots \times (1)$$

$$= \frac{(n+1)!}{(n+1)!}$$
$$= 1 \quad (B)$$

Equating (A) & (B)

$$(n+2)P(n+1) - (n+1) = 1$$
$$(n+2)P(n+1) = 1 + n + 1$$
$$P(n+1) = \frac{n+2}{n+2}$$
$$P(n+1) = 1$$

(c)	(i)	Show that $x - \log_e(1+x) > 0$ for $x > 0$.	2
	(ii)	Hence show that $\sum_{k=1}^{n} \frac{1}{k} > \log_e(n+1)$.	2
	(iii)	Hence by considering $x + \log_e(1-x)$, show that $\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \log_e 2$.	3

i) $$\frac{d}{dx}\left(x - \ln(1+x)\right) = 1 - \frac{1}{1+x}$$
$$= \frac{1+x-1}{1+x}$$
$$= \frac{x}{1+x}$$
$$\geq 0 \quad \text{since } x>0,\ 1+x>0$$

$\therefore (x - \ln(1+x))$ is increasing for all $x>0$.

Also, if $x=0$ $\quad x - \ln(1+x) = 0$

$\therefore \quad x - \log_e(1+x) > 0$ for $x > 0$

ii) using part i)

$$\frac{1}{k} > \ln\left(1+\frac{1}{k}\right) \quad \text{for all } k>0$$

$$\frac{1}{k} > \ln\left(\frac{k+1}{k}\right)$$

$$\therefore \quad \frac{1}{1} > \ln\left(\frac{2}{1}\right) \quad k=1$$

$$\frac{1}{2} > \ln\left(\frac{3}{2}\right) \quad k=2$$

$$\vdots$$

$$\frac{1}{n} > \ln\left(\frac{n+1}{n}\right) \quad k=n.$$

summing

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} > \ln\left(\frac{2}{1}\right)+\ln\left(\frac{3}{2}\right)+\dots+\ln\left(\frac{n+1}{n}\right)$$

$$= \ln\left(\frac{\cancel{2}}{1}\times\frac{\cancel{3}}{\cancel{2}}\times\dots\times\frac{n+1}{\cancel{n}}\right)$$

$$= \ln(n+1)$$

$$\therefore \sum_{k=1}^{n}\frac{1}{k} > \ln(n+1)$$

iii) $$\frac{d}{dx}\left(x+\log_e(1-x)\right) = 1-\frac{1}{1-x}$$

$$= \frac{1-x-1}{1-x}$$

$$= \frac{x}{x-1}$$

For $0<x<1$, $\frac{x}{x-1}<0$ [numerator positive, denominator negative]

$\therefore\ x + \log_e(1-x)$ is decreasing for $0<x<1$

Also, if $x=0$ $\quad x+\log_e(1-x)=0$

$\therefore\quad x+\log_e(1-x)<0 \quad$ for $0<x<1$

and so $\quad \frac{1}{k^2}+\log_e\left(1-\frac{1}{k^2}\right)<0 \quad$ for all $k>1$

$$\therefore \sum_{k=2}^{n} \frac{1}{k^2} < \sum_{k=2}^{n} -\ln\left(1-\frac{1}{k^2}\right)$$

$$= \sum_{k=2}^{n} -\ln\left(\frac{k^2-1}{k^2}\right)$$

$$= \sum_{k=2}^{n} -\ln\left(\frac{(k-1)(k+1)}{k^2}\right)$$

$$= \sum_{k=2}^{n} \ln\left(\frac{k^2}{(k-1)(k+1)}\right)$$

$$= \sum_{k=2}^{n} \left[\ln(k^2) - \ln\left[(k-1)(k+1)\right]\right]$$

$$= \sum_{k=2}^{n} \left(2\ln k - \ln(k-1) - \ln(k+1)\right]$$

$$\therefore \frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2} < \Big[(2\ln 2 - \ln 1 - \cancel{\ln 3}) + (\cancel{2\ln 3} - \ln 2 - \ln 4) + (2\ln 4 - \cancel{\ln 3} - \ln 5) + \dots + 2\ln(n) - \ln(n-1) - \ln(n+1)\Big]$$

$$= \ln 2 + \ln(n) - \ln(n+1)$$

$$1 + \frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2} < 1 + \ln 2 + \ln\left(\frac{n}{n+1}\right)$$

$$\therefore \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \ln 2 + \ln\left(\frac{n}{n+1}\right)$$

As $n\to\infty$, $\frac{n}{n+1}\to 1 \quad \therefore \ln\left(\frac{n}{n+1}\right)\to 0$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2.$$

End of paper